Geometric limits of knot complements, II: Graphs determined by their complements

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March 22, 2011

Abstract

We prove that there are compact submanifolds of the 3–sphere whose interiors are not homeomorphic to any geometric limit of hyperbolic knot complements.

A *hyperbolic knot complement* is a complete hyperbolic 3–manifold homeomorphic to the complement of a knot in $\mathbb{S}^3$. A complete hyperbolic 3–manifold M is a *geometric limit* of hyperbolic knot complements if for every positive ε, every compact submanifold K in M admits a $(1+\varepsilon)$–bilipschitz embedding into a hyperbolic knot complement. Geometric limits of hyperbolic knot complements were studied by J. Purcell and the second author, who proved that every one–ended hyperbolic 3–manifold with finitely generated fundamental group that embeds in $\mathbb{S}^3$ is a geometric limit of hyperbolic knot complements [17, Theorem 1.1]. It follows that every compression body is homeomorphic to a geometric limit of hyperbolic knot complements, and, in particular, there are geometric limits of hyperbolic knot complements with arbitrarily many ends.

The topology of such examples is not limited to that of compression bodies.

Example 1. *There are compact hyperbolic* 3*–manifolds with totally geodesic disconnected boundary whose interior is homeomorphic to a geometric limit of hyperbolic knot complements.*

It should be remarked that by [17, Theorem 1.3], a hyperbolic 3–manifold with at least two convex cocompact ends is not a geometric limit of hyperbolic knot complements. The example shows that such a manifold may nonetheless be homeomorphic to one.

These results lead Purcell and Souto to wonder if a compact submanifold of $\mathbb{S}^3$ whose interior admits a convex cocompact hyperbolic structure could fail to be homeomorphic to any geometric limit of knot complements [17, Question 4]. We construct such a manifold here.

Example 2. *There is a compact submanifold of* $\mathbb{S}^3$ *that admits a hyperbolic metric with totally geodesic boundary whose interior is not homeomorphic to any geometric limit of knot complements.*

*The second author has been partially supported by NSF grant DMS-0706878 and the Alfred P. Sloan Foundation.

Using elementary arguments about geometric limits, we obtain Example 2 as a corollary of the following theorem.

Theorem 3. *There is a compact oriented hyperbolic* 3*–manifold* M *with totally geodesic disconnected boundary that admits a unique orientation preserving embedding* $M \to \mathbb{S}^3$ *up to isotopy. In fact, if* Θ *is a graph, then* Θ *admits an embedding into* $\mathbb{S}^3$ *so that the exterior of* Θ *has a unique oriented embedding into* $\mathbb{S}^3$ *up to isotopy.*

Note that it follows from Fox's reembedding theorem [7] that any compact submanifold of $\mathbb{S}^3$ that admits a single orientation preserving embedding in $\mathbb{S}^3$ is the exterior of an embedded graph.

Using work of M. Lackenby [10], Theorem 3 is reduced to finding embeddings of Θ satisfying a certain geometric condition, see Section 1. These embeddings are obtained in Sections 2 and 3 by a variation of arguments in [9]. Theorem 3 and the examples are established in the final section.

Acknowledgments. The authors thank Jessica Purcell for her interest and encouragement as this work developed, and Jeff Brock, Ken Bromberg, and Dick Canary for nice conversations.

1 Short obvious meridians and unique embeddings

A vertex of a graph is **extremal** if it has valence less than or equal to one. In this section we consider finite embedded graphs $\Theta \subset \mathbb{S}^3$ without extremal vertices. A **branch vertex** is a vertex of valence at least three. We say that Θ is **trivalent** if all of its branch vertices have valence three.

Every component of a regular neighborhood $\mathcal{N}(\Theta)$ of an embedded graph $\Theta \subset \mathbb{S}^3$ is a handlebody. Given an edge e of Θ, we say that an essential simple closed curve $m \subset \partial\mathcal{N}(\Theta)$ is an **obvious meridian** associated to the edge e if m bounds a disk D in $\mathcal{N}(\Theta)$ that intersects Θ transversally in a single point in the interior of e. Since $\mathcal{N}(\Theta) \setminus \Theta$ is homeomorphic to $\partial\mathcal{N}(\Theta) \times [0,1)$, every edge has an obvious meridian and any two obvious meridians for e are isotopic in $\partial\mathcal{N}(\Theta)$. Note that if Θ is trivalent and $\Delta \subset \Theta$ is a connected component that is not a circle, then the obvious meridians form a pants decomposition of $\partial\mathcal{N}(\Delta)$.

Following Myers [12], a compact oriented 3–manifold M is **excellent** if it is irreducible atoroidal and acylindrical. Equivalently, the complement of the torus boundary components of M admits a complete hyperbolic metric with totally geodesic boundary, by Thurston's Geometrization Theorem for Haken Manifolds [14]. By Mostow–Prasad rigidity [11, 16], such a metric is unique up to isometry. Given Θ, we let M_Θ be the closure in $\mathbb{S}^3$ of the complement of $\mathcal{N}(\Theta)$. With Myers, we say that Θ is **excellent** if M_Θ is.

Suppose from now on that M_Θ is excellent and let M'_Θ be the complete hyperbolic manifold with totally geodesic boundary homeomorphic to the complement in M_Θ of the torus boundary components. Every torus T in ∂M_Θ corresponds to a rank–two cusp of M'_Θ. Let $\mathbf{P}(T)$ be the Margulis tube at this cusp with the property that the shortest essential curve in $\partial\mathbf{P}(T)$ has length one. It was proved in [1] that the union $\mathbf{P}$ of these

Margulis tubes over all torus components of ∂M_Θ is a *disjoint* union that does not meet $\partial M'_\Theta$. We identify M_Θ with the complement of $\mathbf{P}$ in M'_Θ. On each component of ∂M_Θ of negative Euler characteristic this determines a hyperbolic metric, and on each torus a flat metric. So we may refer to the **length** $\ell(\gamma)$ of a curve γ in ∂M_Θ without risk of confusion.

Given ε positive, we say that an excellent graph $\Theta \subset \mathbb{S}^3$ has **ε–short obvious meridians** if every obvious meridian m on a torus T in ∂M_Θ has length $\ell(m) < \varepsilon \operatorname{area}(T)$, and every other obvious meridian has $\ell(m) < \varepsilon$.

We will refer to the subgraphs of Θ without extremal vertices as the **descendants** of Θ. Both Θ and $\emptyset$ are descendants of Θ, and every descendant of a trivalent graph is trivalent. Our interest in excellent graphs, their descendants, and short obvious meridians is due to the following proposition.

Proposition 4. *For each n, there is an ε such that when $\Theta \subset \mathbb{S}^3$ is an excellent trivalent graph with $|\chi(\Theta)| \leq n$ such that every nonempty descendant $\Delta \subset \Theta$ is excellent and has ε–short obvious meridians, then every embedding $M_\Theta \to \mathbb{S}^3$ is the restriction of a diffeomorphism $\mathbb{S}^3 \to \mathbb{S}^3$. In particular, M_Θ admits a unique orientation preserving embedding $M_\Theta \to \mathbb{S}^3$ up to isotopy.*

We derive Proposition 4 from the following result of Lackenby [10].

Theorem (Lackenby)**.** *Let M be a compact excellent 3–manifold and let $\mathcal{C}$ be the set of essential simple closed curves $\gamma \subset \partial M$ such that if γ lies in a component S of ∂M with negative Euler characteristic, then*

$$\ell(\gamma) \leq \frac{4\pi}{(1-4/\chi(S))^{1/4} - (1-4/\chi(S))^{-1/4}}$$

and if γ lies in a torus, then $\ell(\gamma) \leq 2\pi$.

Along each component S of ∂M attach a 3–manifold H_S via a homeomorphism $S \to \partial H_S$ to obtain a manifold N. Then either N has infinite fundamental group or there is a component S of ∂M containing a curve $\gamma \in \mathcal{C}$ which is sent by $S \to \partial H_S$ to a curve bounding a properly embedded disk in H_S. □

In [10], this is only stated and proved when the H_S are handlebodies, but the proof applies without this restriction—see the proof of [17, Theorem 1.3]. Proposition 4 follows easily from Lackenby's theorem and the following observation.

Lemma 5. *For each n, there is an ε such that when $\Theta \subset \mathbb{S}^3$ is a trivalent excellent graph with $|\chi(\Theta)| \leq n$ and ε–short obvious meridians, then the collection $\mathcal{C}$ in Lackenby's theorem is a collection of obvious meridians.*

Proof. First consider a curve γ in $\mathcal{C}$ that lies in a component S of ∂M_Θ with $\chi(S) < 0$. Note that $|\chi(S)| \leq 2n$. Suppose that each obvious meridian in S has length no more than ε. By the Collar Lemma, each has a collar of width $\log(\coth(\varepsilon/4))$. In particular, if ε is small enough that

$$\log\left(\coth\left(\frac{\varepsilon}{4}\right)\right) > \frac{4\pi}{(1+4/n)^{1/4} - (1+4/n)^{-1/4}}$$

then γ must be isotopic off of the obvious meridians. Since the obvious meridians in S form a pants decomposition, γ is an obvious meridian itself.

Now consider a torus T in ∂M_Θ and recall that T is endowed with a Euclidean metric of injectivity radius one. Suppose there is a curve γ in T with length $\ell(\gamma) \leq 2\pi$ that is not isotopic to the obvious meridian m in T. Then

$$1 \leq \operatorname{area}(T) \leq \ell(\gamma)\ell(m) \leq 2\pi\ell(m),$$

and this is not possible if Θ has $(1/4\pi)$–short obvious meridians. $\square$

Proof of Proposition 4. Given n, let ε be the number provided by Lemma 5 and suppose that $\Theta \subset \mathbb{S}^3$ is a trivalent graph such that all of its nonempty descendants are excellent and have ε–short obvious meridians.

Let $\varphi\colon M_\Theta \to \mathbb{S}^3$ be an embedding. For every component S of ∂M_Θ let H_S be the connected component of $\mathbb{S}^3 \setminus \varphi(M_\Theta)$ facing $\varphi(S)$. So, gluing the manifolds H_S to N_Θ via the boundary identifications induced by φ, we obtain the 3–sphere. It follows from Lackenby's theorem and Lemma 5 that there is an obvious meridian $m \subset \partial M_\Theta$ whose image under φ bounds a properly embedded disk in H_S. Let e be the edge of Θ corresponding to m and let Θ' be the largest descendant of Θ that does not contain e.

The manifold $M_{\Theta'}$ is homeomorphic to the manifold obtained by attaching a 2–handle to M_Θ along m and capping off boundary spheres. Since the image of m under φ bounds a disk in H_S, the embedding φ extends to an embedding $\varphi'\colon M_{\Theta'} \to \mathbb{S}^3$. If Θ' is empty, then $M_{\Theta'} = \mathbb{S}^3$ and we are done.

Otherwise, observe that $|\chi(\Theta')| \leq |\chi(\Theta)|$. By assumption, Θ' is excellent and has ε–short meridians. The above argument yields a descendant Θ'' of Θ' such that the embedding φ' extends to an embedding $\varphi''\colon N_{\Theta''} \to \mathbb{S}^3$. Repeating this process—no more often than the number of edges in Θ—we obtain a diffeomorphism $\mathbb{S}^3 \to \mathbb{S}^3$ which extends φ.

The final statement follows from the fact that the group of orientation preserving diffeomorphisms of $\mathbb{S}^3$ is connected. $\square$

We now set about constructing graphs with arbitrarily short obvious meridians.

2 Really excellent graphs

Let X be a finite graph. We let ∂X denote the set of extremal vertices in X, and write $X^\circ = X - \partial X$. Let M a 3–manifold. An embedding $X \to M$ is **proper** if it induces a map of triples $(X, X^\circ, \partial X) \to (M, M^\circ, \partial M)$. A finite graph X properly embedded in M is **excellent** if the exterior $M \setminus X$ is excellent. We say that a proper embedding $X \to M$, or its image, is **nice** if X is nonempty, has no isolated vertices, and no component of $\partial(M \setminus X)$ is a sphere. Given a nice proper embedding $X \to M$, a subgraph Y is nice if the induced embedding is.

We will need the following mild generalization of Myers' theorem [12].

Theorem 6. *Let M be an oriented 3–manifold. Let X be a finite graph without isolated vertices. If there is a nice embedding $X \to M$, then there is nice embedding $X \to M$ with the property that $M \setminus Y$ is excellent for any nice subgraph Y.*

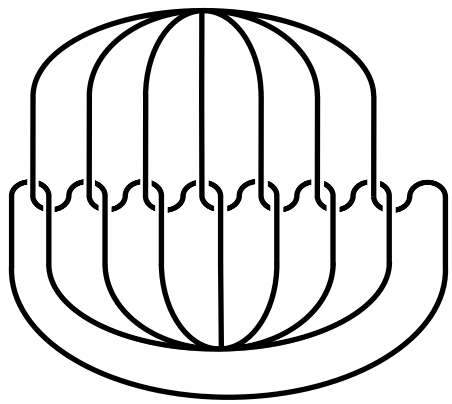

Figure 1: Suzuki's Brunnian graph on seven edges.

Theorem 6 has the following corollary.

Corollary 7. *Let X be a finite graph without isolated vertices properly embedded in a handlebody H of positive genus. Then there is an embedding $X \to H$ with the property that if Y is any nonempty subgraph of X without isolated vertices, then $H \setminus Y$ is excellent.* □

We need some preliminary lemmata. Let $\mathbb{B}$ be a 3–ball. Recall that a tangle is **Brunnian** if removing *any* strand results in the trivial tangle.

Lemma 8. *For each n, there is an excellent Brunnian n–tangle in $\mathbb{B}$.*

Proof. The exterior of Suzuki's Brunnian graph Θ_{n+1} on $n+1$ edges, pictured in Figure 1, admits a hyperbolic structure with totally geodesic boundary, see [15, 21], and is thus excellent. Moreover, the exterior of Θ_{n+1} is homeomorphic to the exterior of a Brunnian n–tangle in a ball—see Figure 2. Since the exterior of Θ_{n+1} is excellent, so is the exterior of the tangle. □

If X is a graph, and Γ a subgraph, we let $X - \Gamma$ be the graph obtained by removing Γ and taking the closure. If X and Y are properly embedded graphs in a 3–manifold M, we write $X \sim Y$ if they are ambiently isotopic.

Lemma 9. *Let $g\colon X \to M$ be a nice embedding. Let Δ be a nice subgraph of X. There is an embedding $f\colon X \to M$ with $f(\Delta)$ excellent and such that for any edge e of Δ,*

$$f(\Delta - e) \sim g(\Delta - e) \quad \text{and} \quad f(X - e) \sim g(X - e).$$

Proof. Let e_1, e_2, …, e_m be the edges of Δ. For each i, let y_i be a point in the interior of e_i and let $x_i = g(y_i)$. Let α be an arc in M whose left endpoint is x_1, whose right endpoint is x_m, and whose intersection with $g(X)$ is $\{x_1, \ldots, x_m\}$.

By Myers' theorem [12], the arc α is homotopic relative to $\{x_1, \ldots, x_m\}$ to an arc β with $\beta \cap g(X) = \{x_1, \ldots, x_m\}$ such that $g(\Delta) \cup \beta$ is excellent.

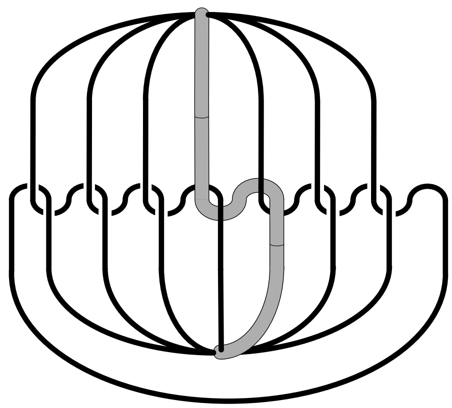

Figure 2: An excellent Brunnian tangle on six strands.

Let $\mathcal{N}(\beta)$ be a regular neighborhood of β whose intersection with $g(X)$ is the standard trivial m–tangle in $\mathcal{N}(\beta) \cong \mathbb{B}$, each strand containing an x_i. It is easy to see that $\partial\mathcal{N}(\beta) \setminus g(\Delta)$ is incompressible in $M \setminus (g(\Delta) \cup \mathcal{N}(\beta))$. Moreover, the exterior of $\mathcal{N}(\beta) \setminus g(\Delta)$ in $M \setminus g(\Delta)$ is excellent, as it is homeomorphic to $M \setminus (g(\Delta) \cup \beta)$.

Now, in the obvious way, replace $\mathcal{N}(\beta) \setminus g(\Delta) = \mathcal{N}(\beta) \setminus g(X)$ with the exterior of an excellent Brunnian m–tangle given by Lemma 8. The resulting manifold is excellent, by Myers' gluing lemma, Lemma 2.1 of [12], and is homeomorphic to the exterior of an embedding $f\colon\ \Delta \to M$. Since the tangle is Brunnian, if e is an edge of Δ, we have

$$f(\Delta - e) \sim g(\Delta - e) \quad \text{and} \quad f(X - e) \sim g(X - e). \qquad \square$$

Lemma 10. *Let X be a finite graph without isolated vertices and let $\mathfrak{S}(X)$ be the set of subgraphs of X without isolated vertices. Then there is an ordering $\Gamma_1, \ldots, \Gamma_m$ of the elements of $\mathfrak{S}(X)$ such that $\Gamma_k \not\subset \Gamma_j$ when $k < j$.*

Proof. If X is a single edge, there is nothing to do.

Let $E > 1$. Suppose that we have proven the lemma for all graphs whose number of edges is strictly less than E, and suppose that X has E edges.

Pick an edge e of X. Partition $\mathfrak{S}(X)$ into two sets $\mathfrak{S}_+$ and $\mathfrak{S}_-$, the first consisting of those subgraphs that contain e, the latter of those that do not. These sets are order isomorphic when equipped with the partial order induced by inclusion, as is easily seen by sending each subgraph in $\mathfrak{S}_-$ to the subgraph in $\mathfrak{S}_+$ obtained by adding e.

By induction, we may order $\mathfrak{S}_-$, and hence $\mathfrak{S}_+$, as desired. Namely

$$\mathfrak{S}_+ = \{\Gamma_1, \ldots, \Gamma_\ell\} \quad \text{and} \quad \mathfrak{S}_- = \{\Gamma_{\ell+1}, \ldots, \Gamma_{2\ell}\}.$$

As no element of $\mathfrak{S}_+$ is contained in any element of $\mathfrak{S}_-$, our desired ordering is $\Gamma_1, \ldots, \Gamma_{2\ell}$. $\square$

Proof of Theorem 6. Let $\Gamma_1, \ldots, \Gamma_N$ be the allowable *removable* subgraphs of X—so each $X - \Gamma_i$ is nice. By Lemma 10, we may assume $\Gamma_k \not\subset \Gamma_j$ when $k < j$, and we do so.

By Myers' theorem [12], there is an embedding $g_1 \colon X \to M$ with image X_1 such that $X_1 - g_1(\Gamma_1)$ is excellent. To see this, note that there is an excellent embedding g_0 of $X - \Gamma_1$ in M. We may then build an embedding of Γ_1 into $M - g_0(X - \Gamma_1)$ to obtain the desired embedding.

We want an embedding $g_N \colon X \to M$ with excellent image X_N so that $X_N - g_N(\Gamma_k)$ is excellent for all k.

Suppose that we have constructed an embedding $g_j \colon X \to M$ with image X_j so that for any $k \leq j$, the graph $X_j - g_j(\Gamma_k)$ is excellent. We will construct an embedding $g_{j+1} \colon X \to M$ with image X_{j+1} so that for any $k \leq j+1$, the graph $X_{j+1} - g_{j+1}(\Gamma_k)$ is excellent.

Apply Lemma 9 to g_j with $\Delta = X - \Gamma_{j+1}$ to obtain an embedding $f_j \colon X \to M$ with image Y_j such that $Y_j - f_j(\Gamma_{j+1})$ is excellent and deleting any edge $f_j(e)$ of $f_j(\Delta) = Y_j - f_j(\Gamma_{j+1})$ from Y_j yields a graph ambiently isotopic to $X_j - g_j(e)$.

If $k < j+1$, then $\Gamma_k \not\subset \Gamma_{j+1}$, and so there is an edge e in Γ_k that lies in $\Delta = X - \Gamma_{j+1}$. So

$$Y_j - f_j(\Gamma_k) \sim X_j - g_j(\Gamma_k)$$

when $k \leq j$, which is excellent by induction.

Now apply Lemma 9 to f_j with $\Delta = X$ to obtain an embedding g_{j+1} with excellent image X_{j+1}. So deleting *any* edge $g_{j+1}(e)$ from X_{j+1} yields a graph ambiently isotopic to $Y_j - f_j(e)$.

So, if $k \leq j$, then $X_{j+1} - g_{j+1}(\Gamma_k)$ has the same exterior as $Y_j - f_j(\Gamma_k)$, which has the same exterior as $X_j - g_j(\Gamma_k)$, which is excellent by induction.

Finally note that $X_{j+1} - g_{j+1}(\Gamma_{j+1}) = Y_j - f_j(\Gamma_{j+1})$ is also excellent.

We are now done by induction. $\square$

3 Graphs with short obvious meridians

Let F be a separating surface of genus at least two in a 3–manifold M. Let Θ be a graph intersecting F transversely in a nonempty set of points such that the components M_1 and M_2 of $M \setminus (F \cup \Theta)$ are excellent. Let P be the union of tori in ∂M_Θ and let $Q_j = P \cap M_j$. We consider (M,P) and the (M_j, Q_j) as pared manifolds. Let $S = F \setminus \Theta$. So $M_\Theta = M_1 \cup_S M_2$. If φ is a pure braid in $\mathrm{Mod}(S)$, let M_Θ^φ be the manifold obtained by gluing the M_j together via φ. Such an M_Θ^φ is the exterior of a different embedding of Θ into M and admits a hyperbolic structure with totally geodesic boundary.

Proposition 11. *Let $\varepsilon > 0$. If φ is pseudo-Anosov, then for all sufficiently large n, the length of each component of ∂S is less than ε in the metric on $M^n = M_\Theta^{\varphi^n}$ with totally geodesic boundary.*

Proof. Equip M^n with its hyperbolic metric with totally geodesic boundary, and let

$$\rho^n \colon \pi_1(M^n) \to \mathrm{PSL}_2\mathbb{C}$$

be the associated holonomy representation. Let $\rho_j^n\colon\ \pi_1(M_j) \to \mathrm{PSL}_2\mathbb{C}$ be the representations induced by the inclusions $M_j \to M^n$. Let α be a simple closed essential and nonperipheral curve in S.

Since M_2 is excellent, the set

$$\mathrm{AH}(M_2) \subset \mathrm{Hom}(\pi_1(M_2), \mathrm{PSL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C}$$

of discrete faithful representations is compact, by [19]. In particular, there is an $L > 0$ such that, for all n, the translation length $\ell(\rho_2^n(\alpha))$ of $\rho_2^n(\alpha)$ acting on $\mathbb{H}^3$ is bounded above by L. It follows that, for all n, the geodesic representative of α in M^n has length bounded by L. Viewing S as a subsurface of ∂M_1, we have

$$\ell\left(\rho_1^n\left(\varphi^{-n}(\alpha)\right)\right) \leq L \text{ for all } n. \tag{1}$$

By [20], every subsequence of the ρ_1^n has a further subsequence that converges algebraically to the holonomy representation of a hyperbolic structure with parabolics at Q_1. Let ρ_1^∞ be such a limit. In the space $\mathbb{PML}(S)$ of projective measured laminations on S, the sequence $\varphi^{-n}(\alpha)$ converges to the unstable lamination λ^- of φ. By (1) and continuity of Thurston's length function [3, Corollary 7.3], the lamination λ^- is not realized in ρ_1^∞. Thurston's compactness theorem for pleated surfaces [5, Theorem 5.2.2] now implies that ρ_1^∞ takes each component of ∂S to a parabolic element. Since this is true for any limit obtained by passing to a subsequence of the ρ_1^n, we have that, for all large n, the representation ρ_1^n carries each component of ∂S to an element with small translation length.

It follows that if β is a component of ∂S that does not lie in Q_1, its length in M^n tends to zero.

Given a subsequence of the $\Gamma_1^n = \rho_1^n(\pi_1(M_1))$, we may always pass to a further subsequence that converges algebraically to some Γ_1^∞ and geometrically to some $\widehat{\Gamma}_1$, see [18, Corollary 9.18] and [8, Proposition 3.8]. The manifold $M_1^\infty = \mathbb{H}^3/\Gamma^\infty$ covers $\widehat{M}_1 = \mathbb{H}^3/\widehat{\Gamma}$. By the above, the manifold M_1^∞ has a degenerate relative end E at S—see Section V of [2].

Geometric convergence of a subsequence of the Γ_1^n implies that we may pass to a further subsequence so that the $\Gamma^n = \rho^n(\pi_1(M^n))$ converge geometrically to a manifold $\widehat{M}$ covered by $\widehat{M}_1$, and hence M_1^∞. To see this, let γ be a closed geodesic in $\widehat{M}_1$ corresponding to a nonperipheral embedded curve in S, and let γ_n be the corresponding geodesics in the $M_1^n = \mathbb{H}^3/\Gamma_1^n$. We choose basepoints x_n in the γ_n to realize the geometric convergence $M_1^n \to \widehat{M}_1$, and let y_n be the image of x_n in M^n. We claim that there is an $r > 0$ such that the injectivity radius of M^n at y_n is at least r. Suppose that this is not the case. Then for each $\varepsilon > 0$, the image of γ_n in M^n intersects the ε–thin part of M^n for infinitely many n. Now, the lengths of the γ_n are uniformly bounded below. Since the M_1 and M_2 are excellent, any embedded curve in S is primitive in $\pi_1(M^n)$, and so γ_n is primitive there. It follows that γ_n cannot lie entirely in an ε–Margulis tube when ε is small compared to the length of γ_n. By Brooks and Matelski's theorem [4], the distance between the boundary of the δ–thin part and that of the ε–thin part tends to infinity as ε tends to zero, and we are forced to conclude that the lengths of the γ_n tend to infinity after passing to a subsequence. But the lengths of the γ_n are bounded above, as they tend to the length of γ.

Since the set of manifolds with injectivity radius $r > 0$ at the basepoint is compact in the geometric topology [5, Corollary 3.1.7], we may pass to a subsequence so that the (M^n, y_n) converge geometrically to a manifold $\widehat{M}$, which is covered by $\widehat{M}_1$.

By the Covering Theorem [6], the restriction of the covering map $M_1^\infty \to \widehat{M}$ to E is finite–to–one. It follows that, in $\widehat{M}$, the cusps corresponding to the parabolic elements at Q_i are rank–one cusps.

So, if β lies in $\partial Q_1 = \partial Q_2$, it has zero extremal length in a cusp cross section of any geometric limit $\widehat{M}$ of the M^n, and so its extremal length tends to zero in the M^n. $\square$

Theorem 12. *Let Θ be a nonempty graph with $\partial\Theta = \emptyset$ and let $\varepsilon > 0$. There is an embedding of Θ into $\mathbb{S}^3$ such that if Δ is a nice subgraph of Θ , then $\mathbb{S}^3 \setminus \Delta$ admits a hyperbolic structure with totally geodesic boundary and ε–short obvious meridians.*

Proof. Let F be a genus–2 Heegaard surface cutting $\mathbb{S}^3$ into handlebodies $\mathbb{A}$ and $\mathbb{B}$. Choose an embedding $\Theta \to \mathbb{S}^3$ such that the vertices of Θ miss F and each edge of Θ intersects F transversally in at least one point. It follows that if Δ is a nonempty connected descendant of Θ, then each edge of Δ intersects F.

The surface F cuts Θ into two properly embedded graphs $X \subset \mathbb{A}$ and $Y \subset \mathbb{B}$. Let $X \to \mathbb{A}$ and $Y \to \mathbb{B}$ be the embeddings given by Corollary 7, and call the images X and Y again.

Let $S = F - X = F - Y$, a punctured surface of genus two. Given a pure braid ζ in $\mathrm{Mod}(S)$, we obtain a new embedding $\Theta \to \mathbb{S}^3$ by realizing ζ as a homeomorphism h of F fixing $\Theta \cap F$ pointwise, cutting $\mathbb{S}^3$ along F, and regluing via h.

Let $\Delta_1, \ldots, \Delta_n$ be the nice subgraphs of Θ. By reversing the order given by Lemma 10, we may assume that if Δ_j is a subgraph of Δ_i, then $j < i$. In particular, $\Delta_n = \Theta$.

For each i, let $S_i = F - \Delta_i$. Each S_i is a punctured surface of genus two, and so admits a Brunnian pseudo-Anosov braid φ_i, compare [22]—a mapping class is **Brunnian** if it becomes the identity *whenever* a puncture is filled.

If ψ is a pure braid in $\mathrm{Mod}(S)$, we let $\kappa_j(\psi)$ denote the descendant of ψ in $\mathrm{Mod}(S_j)$. For each j, we choose a braid ψ_j in $\mathrm{Mod}(S)$ with $\kappa_j(\psi_j) = \varphi_j$ and such that if Δ_j is not a subgraph of Δ_i, then $\kappa_i(\psi_j) = 1$. This is possible as we may choose ψ_j to be a braid descending to φ_j such that filling any puncture of S that survives in S_j kills ψ_j.

If $a_1, \ldots, a_{i-1}$ are integers, then, for all sufficiently large N, the pure braid

$$\varphi_i^N \kappa_i(\psi_{i-1}^{a_{i-1}}) \cdots \kappa_i(\psi_1^{a_1})$$

in $\mathrm{Mod}(S_i)$ is pseudo-Anosov.

Let

$$\zeta_\ell = \psi_n^{\ell_n} \cdots \psi_2^{\ell_2} \psi_1^{\ell_1},$$

and let $\zeta_{\ell,i} = \kappa_i(\zeta_\ell)$. By our ordering of the Δ_j and choice of the ψ_j,

$$\begin{aligned} \zeta_{\ell,i} &= \kappa_i(\psi_n^{\ell_n}) \cdots \kappa_i(\psi_1^{\ell_1}) \\ &= \varphi_i^{\ell_i} \kappa_i(\psi_{i-1}^{\ell_{i-1}}) \cdots \kappa_i(\psi_1^{\ell_1}) \end{aligned}$$

Now, by Proposition 11, we may choose $\ell_n \gg \ell_{n-1} \gg \cdots \gg \ell_1 \gg 1$ and $\mathbb{L} \gg \ell_n$ so that all of the gluing maps $\zeta_{\ell,i}^{\mathbb{L}} \colon S_i \to S_i$ yield graphs Θ_i in $\mathbb{S}^3$ whose complements admit hyperbolic structures with totally geodesic boundary in which the obvious meridians all have length less than ε. So Θ_n is the image of the desired embedding. $\square$

4 Manifolds inaccessible to knot complements

Proof of Theorem 3. Let Θ be a graph and let Φ be the union of the tree components of Θ. A regular neighborhood of Φ is a disjoint union of balls, and it is easy to see that $\mathbb{S}^3 \setminus \Theta$ has a unique oriented embedding into $\mathbb{S}^3$ if $\mathbb{S}^3 \setminus (\Theta - \Phi)$ does.

So let Θ be a graph with no tree components. If Θ is empty, there is nothing to do. Assume that Θ is nonempty. Every graph exterior in $\mathbb{S}^3$ is homeomorphic to the exterior of a 3–valent graph, and so we assume that Θ is 3–valent. Since no component of Θ is a tree, its exterior is homeomorphic to the exterior of a 3–valent graph without extremal vertices, and we assume that Θ has this property as well.

Let ε be the constant provided by Theorem 4 when $n = |\chi(\Theta)|$. By Theorem 12, there is an embedding $\iota\colon \Theta \to \mathbb{S}^3$ such that every nonempty descendant of $\iota(\Theta)$ is excellent and has ε–short obvious meridians. Let $M = \mathbb{S}^3 \setminus \mathcal{N}(\iota(\Theta))$ be the exterior of a regular neighborhood of $\iota(\Theta)$ in $\mathbb{S}^3$. Since $\iota(\Theta)$ is excellent, M admits a hyperbolic metric with totally geodesic boundary. By Theorem 4, any two orientation preserving embeddings $M \to \mathbb{S}^3$ are isotopic. $\square$

We are now ready to produce the Examples from the introduction.

Example 2. Let $N \subset \mathbb{S}^3$ be a manifold with disconnected boundary containing no tori or spheres provided by Theorem 3.

Suppose there is a sequence of hyperbolic knot complements $M_i = \mathbb{S}^3 \setminus K_i$ which converges geometrically to a complete hyperbolic manifold M homeomorphic to the interior of N. Identify N with a compact submanifold of M in such a way that $M \setminus N$ has a product structure. Geometric convergence provides a sequence of better and better bilipschitz embeddings $\varphi_i\colon N \to M_i$. The M_i are knot complements and so sit naturally in the 3–sphere. Composing the φ_i with these natural embeddings $M_i \to \mathbb{S}^3$ we obtain embeddings $\psi_i\colon N \to \mathbb{S}^3$. By Theorem 3, we may postcompose the maps $M_i \to \mathbb{S}^3$ with homeomorphisms of $\mathbb{S}^3$ so that each ψ_i is the identity map. It follows that, for all i, the knot K_i is disjoint from N.

Since ∂N is disconnected, $\mathbb{S}^3 \setminus N$ has at least two components. Observe that by construction all these components are handlebodies. Passing to a subsequence we may assume that all of the K_i lie in a single one of these, U say. Letting V be a component of $\mathbb{S}^3 \setminus N$ different from U, we see that if $m \subset \partial N$ bounds a disk embedded in V then the curves $\varphi_i(m)$ are homotopically trivial in the M_i for all i.

Now, the φ_i are bilipschitz embeddings. So there is an $L > 1$ such that the curves $\varphi_i(m)$ have length at most L for all i. A homotopically trivial curve of length L in a hyperbolic 3–manifold bounds a disk with diameter no more than L. It follows that the curve m is homotopically trivial in the geometric limit M. But the inclusion $N \to M$ is a homotopy equivalence and m is essential in N. $\square$

Example 1. Consider the genus–2 Heegaard splitting $\mathbb{S}^3 = H \cup H'$ and let Θ be an excellent graph in H. Let $\varphi\colon \partial H \to \partial H$ be a pseudo-Anosov mapping class which extends to H and whose attracting and repelling laminations lie in the Masur domain of H', and consider the sequence of manifolds N_n obtained by gluing $H \setminus \Theta$ and H' via φ^n. Note that N_n is homeomorphic to $\mathbb{S}^3 \setminus \varphi^n(\Theta)$. As in [13], it follows that for all sufficiently large n, the manifold N_n is excellent and hence admits a hyperbolic metric with totally geodesic boundary. Equip N_n with this metric. It is an (advanced) exercise in Kleinian groups to prove that the sequence N_n has a geometric limit N_∞ homeomorphic to the interior of $H \setminus \Theta$. Furthermore, the manifold N_n satisfies the conditions of [17, Theorem 1.1] and is hence a geometric limit of knot complements. Being a geometric limit of geometric limits of knot complements, N_∞ is a geometric limit of knot complements itself. □

References

[1] Steven A. Bleiler and Craig D. Hodgson. Spherical space forms and Dehn filling. *Topology*, 35(3):809–833, 1996.

[2] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, 124(1):71–158, 1986.

[3] Jeffrey F Brock. Continuity of thurston's length function. *Geometric And Functional Analysis*, 10(4):741–797, 2000.

[4] Robert Brooks and J. Peter Matelski. Collars in Kleinian groups. *Duke Math. J.*, 49(1):163–182, 1982.

[5] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on notes of Thurston. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 3–92. Cambridge Univ. Press, Cambridge, 1987.

[6] Richard D. Canary. A covering theorem for hyperbolic 3-manifolds and its applications. *Topology*, 35(3):751–778, 1996.

[7] Ralph H. Fox. On the imbedding of polyhedra in 3-space. *Ann. of Math. (2)*, 49:462–470, 1948.

[8] Troels Jørgensen and Albert Marden. Algebraic and geometric convergence of Kleinian groups. *Math. Scand.*, 66(1):47–72, 1990.

[9] Richard P. Kent IV. Totally geodesic boundaries of knot complements. *Proc. Amer. Math. Soc.*, 133(12):3735–3744 (electronic), 2005.

[10] Marc Lackenby. Attaching handlebodies to 3-manifolds. *Geom. Topol.*, 6:889–904 (electronic), 2002.

[11] G. D. Mostow. On the rigidity of hyperbolic space forms under quasiconformal mappings. *Proc. Nat. Acad. Sci. U.S.A.*, 57:211–215, 1967.

[12] Robert Myers. Excellent 1-manifolds in compact 3-manifolds. *Topology Appl.*, 49(2):115–127, 1993.

[13] Hossein Namazi and Juan Souto. Heegaard splittings and pseudo-Anosov maps. *Geom. Funct. Anal.*, 19(4):1195–1228, 2009.

[14] Jean-Pierre Otal. Thurston's hyperbolization of Haken manifolds. In *Surveys in differential geometry, Vol. III (Cambridge, MA, 1996)*, pages 77–194. Int. Press, Boston, MA, 1998.

[15] Luisa Paoluzzi and Bruno Zimmermann. On a class of hyperbolic 3-manifolds and groups with one defining relation. *Geom. Dedicata*, 60(2):113–123, 1996.

[16] Gopal Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.*, 21:255–286, 1973.

[17] Jessica S. Purcell and Juan Souto. Geometric limits of knot complements. *J. Topology*, 3(4):759–785, 2010.

[18] William P. Thurston. The geometry and topology of 3–manifolds. Princeton Lecture Notes, 1979. `http://msri.org/publications/books/gt3m`.

[19] William P. Thurston. Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. *Ann. of Math. (2)*, 124(2):203–246, 1986.

[20] William P. Thurston. Hyperbolic Structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary, 1998. `arXiv:math.GT/9801058`.

[21] Akira Ushijima. The canonical decompositions of some family of compact orientable hyperbolic 3-manifolds with totally geodesic boundary. *Geom. Dedicata*, 78(1):21–47, 1999.

[22] Kim Whittlesey. Normal all pseudo-Anosov subgroups of mapping class groups. *Geom. Topol.*, 4:293–307 (electronic), 2000.

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